AUTOMORPHISM GROUPS OF NILPOTENT GROUPS AND LIE ALGEBRAS

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It is not easy to produce many examples of finite p-groups, whose automorphim group is still a p-group, and therefore it is quite surprising that it is proved that for *almost every finite p-group*, the automorphism group is still a p-group. This statement must of course be made more precise but, roughly speaking, it tells that *having* $\mathrm{Aut}(G)$ *a p-group*, is kind of generic situation.

After we became aware of this result, we started wondering whether a similar phenomena appears also in the class of infinite nilpotent groups. The first problem we had to face was how the property *"*$\mathrm{Aut}(G)$ *is a p-group"* should be translated, when infinite groups are concerned.

Once this first fact is settled, a possibly more important detail had to be taken into account, namely how we should spell, in the infinite group setting, the fact that a certain phenomena happens *almost always*.

In this talk I will, first of all, discuss and try to motivate the way we handled these preliminary problems. In our investigation we decided to study nilpotent groups G for which $\mathrm{Aut}(G)$ is nilpotent, and focused our attention on the so called *divisible groups of finite rank*. This class is of particular interest for several reasons. One of them is the existence of the *Mal'cev correspondence* that allows to translate our problem (and many others) into the same problem for nilpotent $\mathbb{Q}$-Lie algebras. The advantage of this new framework, is that tools and viewpoints from algebraic geometry naturally come into the stage. E.g. we prove that the moduli space of isomorphism classes of nilpotent Lie algebras (over a fixed field of characteristic 0) of fixed dimension, is a quotient of an algebraic variety modulo the action of an algebraic group. In such spaces open sets are rather large, since they are dense in any component they intersect, and this suggested what we should have tried to prove. What it turns out is that

- if $d \geq 3$ is fixed,
- $\mathbb{F}$ is a field of characteristic 0 and
- m is a *sufficiently large integer*

the set of isomorphism classes of nilpotent Lie algebras L over $\mathbb{F}$ of class at least m, such that L/L' has dimension d and $\mathrm{Aut}(L)^\circ$ is unipotent, contains a nonempty open subset. The group $\mathrm{Aut}(L)$ is an algebraic group, and $\mathrm{Aut}(L)^\circ$ indicates the connected component of the identity, which is a normal subgroup of finite index.